

Closing Wed: HW_7A,7B (7.5, 7.7, 7.8)

Note: Exam 2 is **Thursday!!!**

Covers 6.4, 6.5, 7.1-7.5, 7.7, 7.8

The exam will roughly look like this:

First 3 pages: 6 integrals (ALL types)

4th page: 6.5, 7.7 and/or 7.8

5th page: 6.4

(8.1 Arc Length is NOT on our midterm)

A Brief Exam 2 Review

Integration!

Work.

Average Value.

Trapezoid, Simpson Rules.

Improper Integrals.

1. The 4 special methods

By parts:

$$xe^{3x}, x^2 \cos(5x),$$
$$\frac{\ln(x-1)}{x^2}, x \tan^{-1}(x).$$

Trig:

$$\sin^3(x) \cos(x), \cos^4(x),$$
$$\sec^3(2x) \tan^3(2x)$$

Trig sub:

$$\frac{1}{x\sqrt{x^2-9}}, \frac{1}{(4-x^2)^{3/2}},$$
$$\sqrt{x^2 + 6x + 10}.$$

Part Frac:

$$\frac{x+2}{(x-1)(x-3)}, \frac{4x}{(x-1)^2(x-3)},$$
$$\frac{5}{x(x^2+4)}, \frac{x^2}{x+7}, \frac{x-3}{x^2+8x+20}$$

2. Substitution and Simplifying

Try $u = \sqrt{x}$, $u = \text{inside}$, $u = e^x$, $u = \text{trig}$.

Completing the square.

Trig facts.

Square identities, half-angle.

Triangle trick.

Random Integrals from Old Finals:

$$1. \int \frac{1-x}{\sqrt{1-x^2}} dx \rightarrow \int \frac{1}{\sqrt{1-x^2}} dx - \int \frac{x}{\sqrt{1-x^2}} dx$$

↳ or $x = \sin \theta$

$$2. \int \frac{x^2 - x + 8}{x^3 + 4x} dx = \int \frac{x^2 - x + 8}{x(x^2 + 4)} dx$$

$$\frac{x^2 - x + 8}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

$$3. \int 2x \ln(x+5) dx$$

$$u = \ln(x+5) \quad dv = 2x dx$$

$$du = \frac{1}{x+5} dx \quad v = x^2$$

$$4. \int \cos^3(x) dx = \int (1 - \sin^2(x)) \cos(x) dx$$

$u = \sin(x)$

$$5. \int_0^2 \frac{1}{\sqrt{x^2 + 2x + 4}} dx = \int_0^2 \frac{1}{\sqrt{(x+1)^2 + 3}} dx$$

$x+1 = \sqrt{3} \tan \theta$

$$6. \int_1^3 \frac{1}{x^2 + x^3} dx = \int_1^3 \frac{1}{x^2(1+x)} dx$$

$\frac{1}{x^2(1+x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{1+x}$

$$7. \int \tan^2 x \sec^4(x) dx = \int \tan^2(x) (1 + \tan^2(x)) \sec^2(x) dx$$

$u = \tan(x)$

$$8. \int \frac{1}{(1 + \sqrt{x})^3} dx = \int \frac{1}{t^3} 2(t-1) dt$$

$= 2 \int t^{-2} - t^{-3} dt$

$$t = 1 + \sqrt{x} \Rightarrow (t-1)^2 = x$$

$2(t-1) dt = dx$

$$9. \int \sin(x) \sqrt{\cos(x)} dx = \int -\sqrt{t} dt$$

$$t = \cos(x)$$

$$dt = -\sin(x) dx$$

$$(-1) dt = \sin(x) dx$$

$$= -\frac{2}{3} t^{3/2} + C$$

$$= -\frac{2}{3} \cos^{3/2}(x) + C$$

3. Improper Integrals:

- Rewrite as a limit!!
- Integrate
- Take limit

Random Improper Integrals:

$$1. \int_1^2 \frac{x}{\sqrt{x-1}} dx = \lim_{t \rightarrow 1^+} \int_t^2 \frac{x}{\sqrt{x-1}} dx$$

$$2. \int_{-\infty}^0 xe^{-x} dx = \lim_{t \rightarrow -\infty} \int_t^0 xe^{-x} dx$$

$$3. \int_1^{\infty} \frac{1}{\sqrt{x}(1+x)} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{\sqrt{x}(1+x)} dx$$

$u = \sqrt{x}$

4. Trapezoid/Simpson Rules

- Set up integral, then compute width and label tickmarks.
- Use formula.

Approximation Example:

- Use Simpson's Method with $n = 4$ subdivision to approximate the value of

$$\int_0^4 \sqrt{1+4x^4} dx$$

$$\Delta x = \frac{4-0}{4} = 1$$

$$\begin{aligned} x_0 &= 0 \\ x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 3 \\ x_4 &= 4 \end{aligned}$$

$$\frac{1}{3} (1) \left[\sqrt{1+4(0)^4} + 4\sqrt{1+4(1)^4} + 2\sqrt{1+4(2)^4} + 4\sqrt{1+4(3)^4} + \sqrt{1+4(4)^4} \right]$$

5. New Applications

a) **Average value** = $\frac{1}{b-a} \int_a^b f(x) dx$

b) **Work** = $\int_a^b (Force)(Dist)$

Step 1: Draw picture (start and end)
Label clearly.

Draw a typical subdivision.

Step 2: Find pattern for *Force* and *Dist*.

Step 3: Integrate.

Type 1: “Changing force”

Force changing as object is moved
(leaky bucket, springs, given force).

$f(x)$ = “force formula at x ”

$Force = f(x)$, $Dist = \Delta x$;

$$Work = \int_a^b f(x) dx$$

Type 2: “Stack of books”

(chain, pumping)

Chain/Cable:

Given k = density = force/length

if $x = 0$ is labeled at the top.

then for any subdivision that makes
it to the top:

$Force = k \Delta x$, $Dist = x$

$$Work = \int_a^b k x dx$$

Pumping:

Given k = density = force/volume

if top is $y = b$, then

$Force = k(\text{Area})\Delta y$, $Dist = b - y$;

$$Work = \int_a^b k(\text{Area})(b - y) dy$$

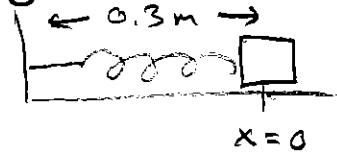
You need to have an basic understanding of
how we label and find pattern to be able to
adapt to changes in these problems!!

Applications from old tests:

1. Find the average value of $\cos^3(x)$ on the interval 0 to $\pi/2$.

$$\frac{1}{\pi/2 - 0} \int_0^{\pi/2} \cos^3(x) dx$$

2. A spring has natural length of 30 cm from the wall. It requires 2 J of work to stretch it from 40 cm to 45 cm (from the wall). How far beyond its natural length will a force of 64 N keep the spring stretched?



$$F_{\text{spring}} = f(x) = kx \quad \text{N}$$

$$W_{\text{work}} = \int_a^b kx dx \quad \text{J}$$

$$0.3 \text{ m} \Rightarrow x=0$$

$$0.4 \text{ m} \Rightarrow x=0.1 \quad \leftarrow (0.4-0.3)$$

$$0.45 \text{ m} \Rightarrow x=0.15 \quad \leftarrow (0.45-0.3)$$

GIVEN

$$\int_{0.1}^{0.15} kx dx = 2$$

$$\Rightarrow \frac{k}{2} x^2 \Big|_{0.1}^{0.15} = 2$$

$$\Rightarrow \frac{k}{2} (0.15^2 - 0.1^2) = 2$$

$$\Rightarrow \frac{k}{2} (0.0125) = 2 \Rightarrow k = 320$$

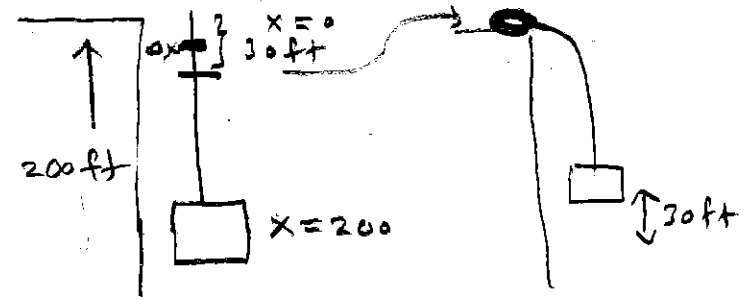
WANT $x = ???$ WHEN FORCE = 64 N

$$\Rightarrow kx = 64$$

$$\Rightarrow x = \frac{64}{320} = \frac{1}{5} = 0.2 \text{ m}$$

20 cm

3. A 1600 lb elevator is suspended by a 200 ft cable that weighs 10 lb/ft. How much work is required to raise the elevator from the basement to the third floor, a distance of 30 ft?



WORK TO LIFT ELEVATOR

$$\text{Force} = 1600 \text{ lbs}$$

$$\text{Dist} = 30 \text{ ft}$$

$$\text{Work} = 1600 \cdot 30$$

$$= 48,000 \text{ ft-lbs}$$

WORK TO LIFT CABLE FOR $0 \leq x \leq 30$

Upper 30 ft of cable makes it to top

$$\text{Force} = 10 \Delta x$$

$$\text{DIST} = x$$

$$\int_0^{30} x \cdot 10 dx$$

$$5x^2 \Big|_0^{30}$$

$$5(30)^2 = 4500 \text{ ft-lbs}$$

NOTE THE DIFFERENCE!

START

WORK TO LIFT CABLE FOR $30 \leq x \leq 200$

ALL OF THIS SHIFTS UP 30 ft

$$\text{Force} = 10 \Delta x$$

$$\text{DIST} = 30$$

$$\int_{30}^{200} 30 \cdot 10 dx$$

$$300x \Big|_{30}^{200}$$

$$300 \cdot (200 - 30)$$

$$300 \cdot 170 = 51,000 \text{ ft-lbs}$$

$$\text{TOTAL} = 48,000 + 4,500 + 51,000 = \boxed{103,500 \text{ ft-lbs}}$$

4. A rope is used to pull a bucket full of water up from a well that is 10 m deep. The rope has a total mass of 5 kg. The bucket has a mass of 11 kg. Find the total work done in lifting the bucket to the top (Recall: Accel. due to gravity is 9.8 m/s^2)

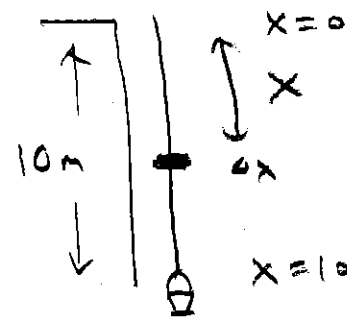
$$\text{MASS} = 11 \text{ kg} \Rightarrow \text{FORCE} = 11 \cdot 9.8 \text{ N} = 107.8 \text{ N}$$

BUCKET

$$\text{FORCE} = 107.8 \text{ N}$$

$$\text{DIST} = 10 \text{ m}$$

$$\Rightarrow \text{WORK} = 107.8 \cdot 10 = 1,078 \text{ J}$$



START



END

ROPE

$$\text{FOR } 0 \leq x \leq 10$$

$$\text{FORCE} = 4.90x$$

$$\text{DIST} = x$$

$$\int_0^{10} x \cdot 4.9 dx$$

$$\frac{4.9}{2} x^2 \Big|_0^{10}$$

$$\frac{4.9}{2} \cdot 100 = 245 \text{ J}$$

$$\text{TOTAL MASS} = 5 \text{ kg} \Rightarrow \text{TOTAL FORCE} = 5 \cdot 9.8 = 49 \text{ N}$$

$$\Rightarrow \text{DENSITY} = \frac{49 \text{ N}}{10 \text{ m}} = 4.9 \frac{\text{N}}{\text{m}}$$

$$\text{TOTAL WORK} = 1078 + 245 = \boxed{1323 \text{ J}}$$

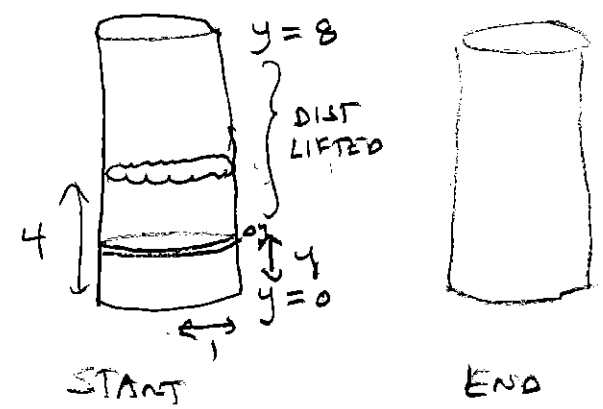
5. A well is in the shape of a cylinder of radius 1 meter and depth 8 meters. It is half full of water. Find the work to pump all the water to the top. (Recall: Water weighs 9800 N/m^3)

For $0 \leq y \leq 4$

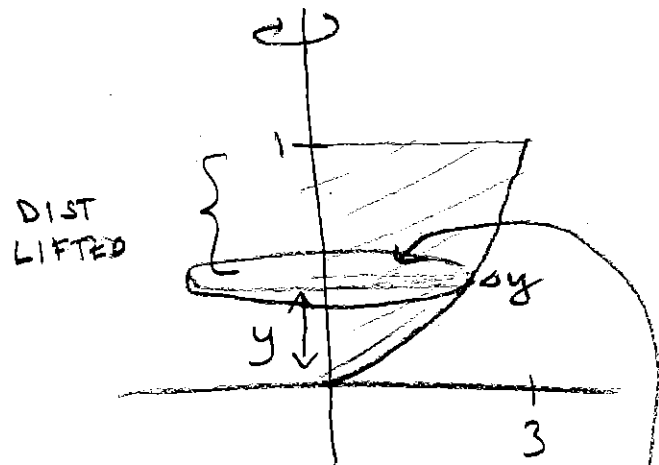
$$\begin{aligned} \text{FORCE} &= 9800 \cdot \overset{\text{HORIZ.}}{\text{AREA}} \cdot dy \\ &= 9800 \cdot \pi (1)^2 dy = 9800\pi dy \end{aligned}$$

$$\text{DIST} = 8 - y$$

$$\begin{aligned} \text{WORK} &= \int_0^4 (8-y) 9800\pi dy \\ &= 9800\pi \left[8y - \frac{1}{2}y^2 \right]_0^4 \\ &= 9800\pi [32 - 8] \\ &= 235,200\pi \text{ J} \approx 738,902.59 \text{ J} \end{aligned}$$



6. The portion of the graph $y = x^2 / 9$ between $x = 0$ and $x = 3$ is rotated about the y -axis to form a container. The container is full of a liquid that has density 100 lbs/ft^3 . Find the work required to pump all the liquid to the top of the container.



For $0 \leq y \leq 1$

$$\text{FORCE} = 100 \cdot \overset{\text{HORIZ.}}{\text{AREA}} \Delta y$$

$$= 100 \pi (3\sqrt{y})^2 \Delta y$$

$$= 900\pi y \Delta y$$

$$\text{DIST} = 1 - y$$

$$\Rightarrow \text{work} = \int_0^1 (1-y) 900\pi y dy = 900\pi \int_0^1 y - y^2 dy$$

$$= 900\pi \left[\frac{1}{2}y^2 - \frac{1}{3}y^3 \right]_0^1$$

$$= 900\pi \left[\frac{1}{2} - \frac{1}{3} \right] = \frac{900\pi}{6}$$

$$= 150\pi \text{ ft-lbs}$$

$$\approx 471.24 \text{ ft-lbs}$$

$$y = \frac{1}{9} x^2$$

$$\Rightarrow 9y = x^2$$

$$\Rightarrow x = 3\sqrt{y}$$